SEMESTRAL EXAMINATION EXAMINATION
M. MATH II YEAR
FOURIER ANALYSIS
I SEMSTER, 2014-15

The 7 questions below carry a total of 110 marks. Answer as many questions as you can. The maximum you can score is 100. Time limit is 3 hours.

1. Prove that $f \in L^1(\mathbb{R}), \hat{f} \in L^1(\mathbb{R}) \Rightarrow f \in L^2(\mathbb{R})$. [10]

2. Let $f(x) = \frac{x}{2}$ for $-\pi < x < \pi, 0$ for $x = \pm\pi$. Write down the Fourier series of f and use it to show (using an appropriate theorem) that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. [15]

3. Let $f(x) = 1$ for $0 < x < \pi, f(x) = -1$ for $-\pi < x < 0, 0$ for $x = \pm\pi$. Find the (discrete) Hilbert transform of f. [15]
[Your answer may be in the form of an infinte sum; it need not be in closed form].

4. If $f, g \in L^1(\mathbb{R}), f \geq 0, g \geq 0$ and $f = f * g$ prove that $f = 0$ a.e. [20]
Hint: deduce that $\int (1 - \cos(tx))g(x)dx = 0$ for $|t|$ sufficiently small unless $f = 0$ a.e.

5. Compute the Fourier transform of $x^2 e^{-x^2/2}$. [10]

6. Show that if $f \in L^1(\mathbb{R})$ and $\int x^2 \left|\hat{f}(x)\right| dx < \infty$ then f is twice continuously differentiable. [20]

7. Let $\Phi = I_{[0,1)}$. Prove that $\sum_{n=-\infty}^{\infty} \left|\hat{\Phi}(t - 2\pi n)\right|^2 = \frac{1}{2\pi}$ a.e. and hence prove that $\sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi t)}{(t+n)^2}$ is a constant. Compute this constant. [20]